The evolution of the concept of proof trough realizability

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October 28, 2021

What is a mathematical proof?

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Proofs are algorithms

What is a mathematical proof?

Proofs are algorithms : realizability

from constructive mathematics to the axiom of choice

Kleene

BHK interpretation

Proofs as functions : ex. a proof of $P \to Q$ is a function f that converts a proof of P into a proof of Q

Computability theory

The study of *computable* functions

Kleene 1945

Proofs are *computable functions* $\to$ Development of Realisability: interpretation of Heyting arithmetic by sets of (indexes) of recursive functions

Realisability

Goal:

To extract the computational content of mathematical proofs. A theory/logical system is interpreted in a model of computation by establishing a correspondence between formulae and programs in a way that is compatible with the rules of deduction.

Exemple: a realizer of $A \Rightarrow B$ is a program which applied to a realizer of A returns a realizer of B

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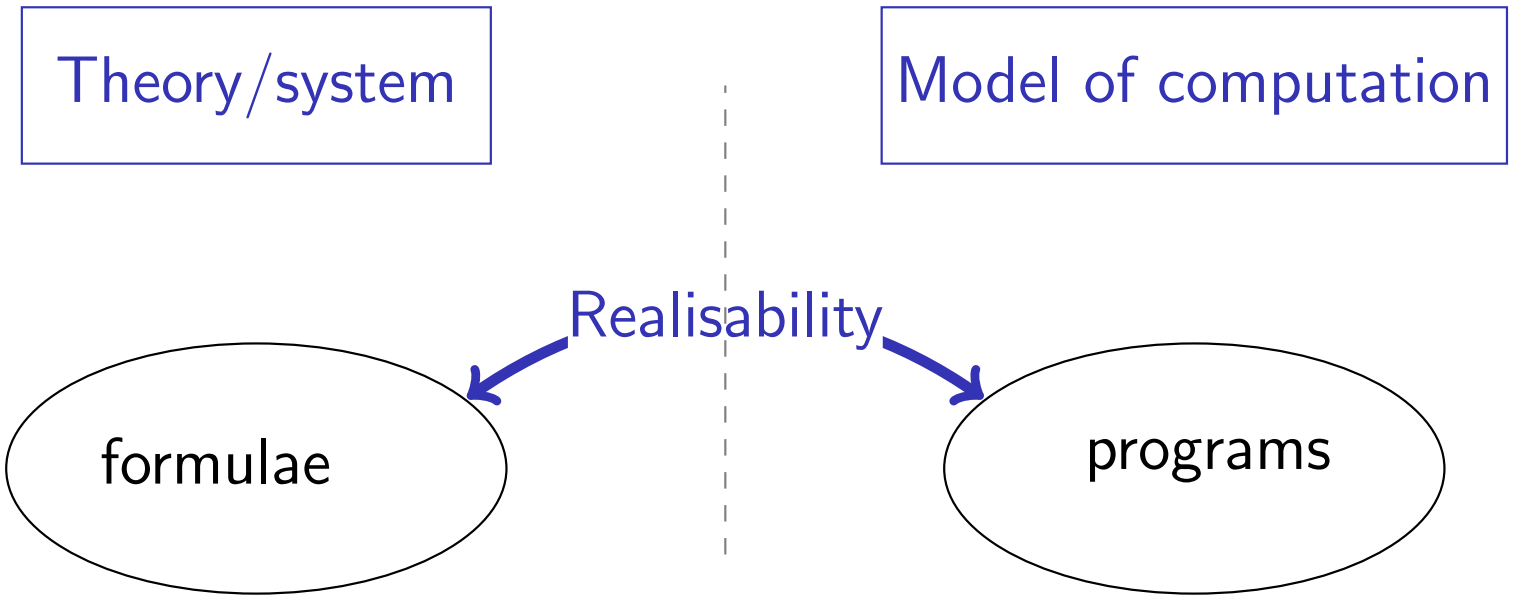
Programs extraction

When a formula A is provable it is possible to extract from every proof of A, a program that realizes it (analogous to programs-extraction in proofs assistants such as Coq).

From Curry Howard to classical realizability



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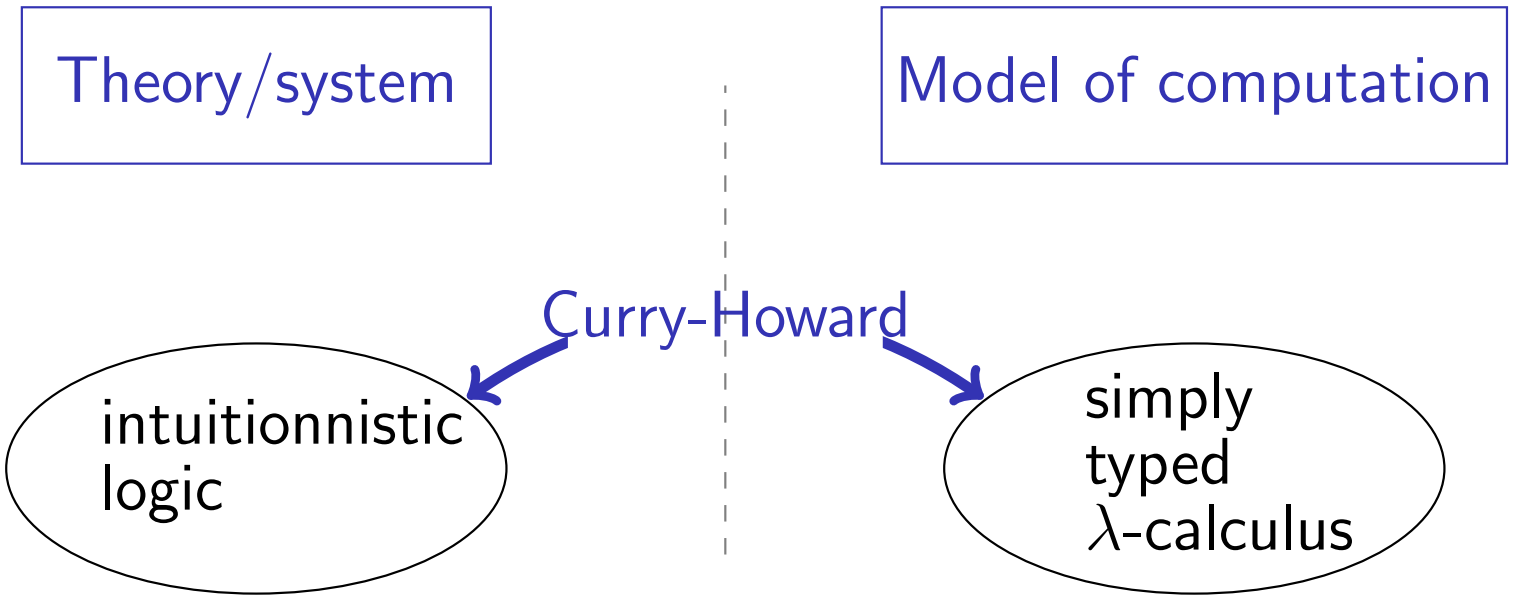

Curry-Howard isomorphism

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow A,\ C \Rightarrow B,\ C \vdash B \Rightarrow A}}{B \Rightarrow A,\ C \Rightarrow B,\ C \vdash A}}{B \Rightarrow A,\ C \Rightarrow B \vdash C \Rightarrow A}}{B \Rightarrow A \vdash (C \Rightarrow B) \Rightarrow C \Rightarrow A}}{\vdash (B \Rightarrow A) \Rightarrow (C \Rightarrow B) \Rightarrow C \Rightarrow A}}{B \Rightarrow A,\ C \Rightarrow B,\ C \vdash C \Rightarrow B \qquad B \Rightarrow A,\ C \Rightarrow B,\ C \vdash C}}{B \Rightarrow A,\ C \Rightarrow B,\ C \vdash B}$$

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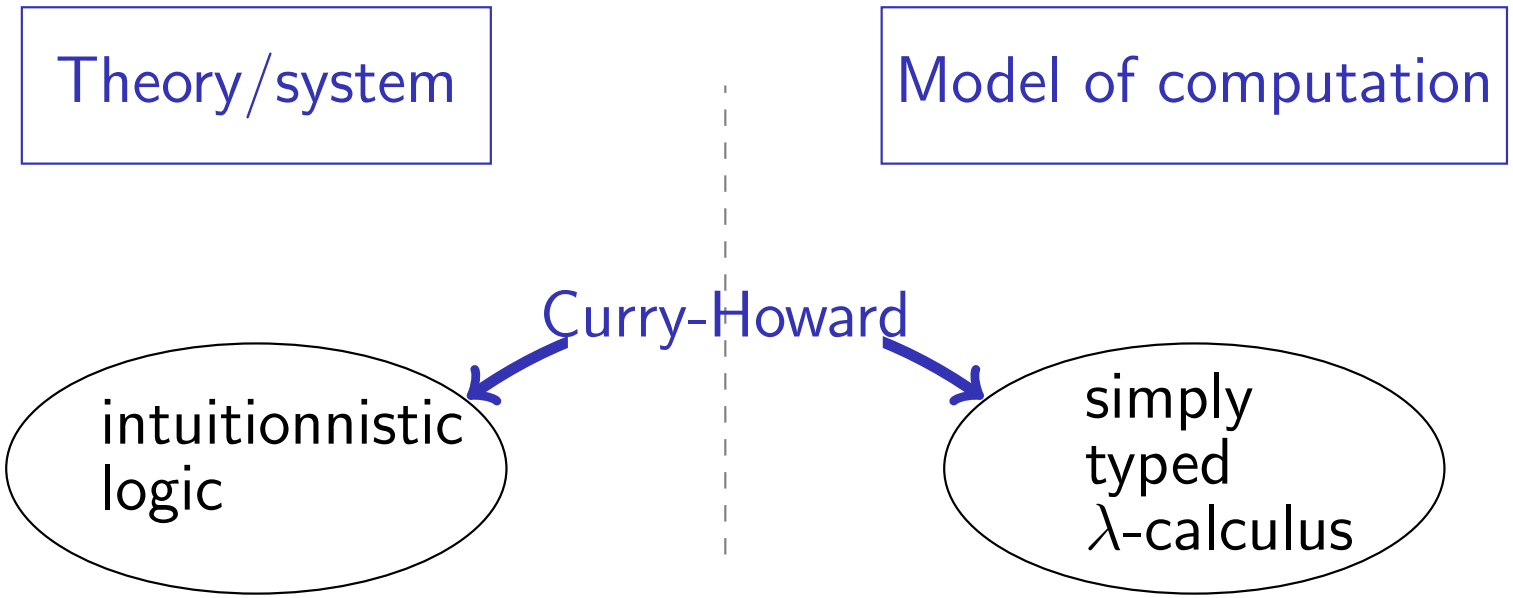

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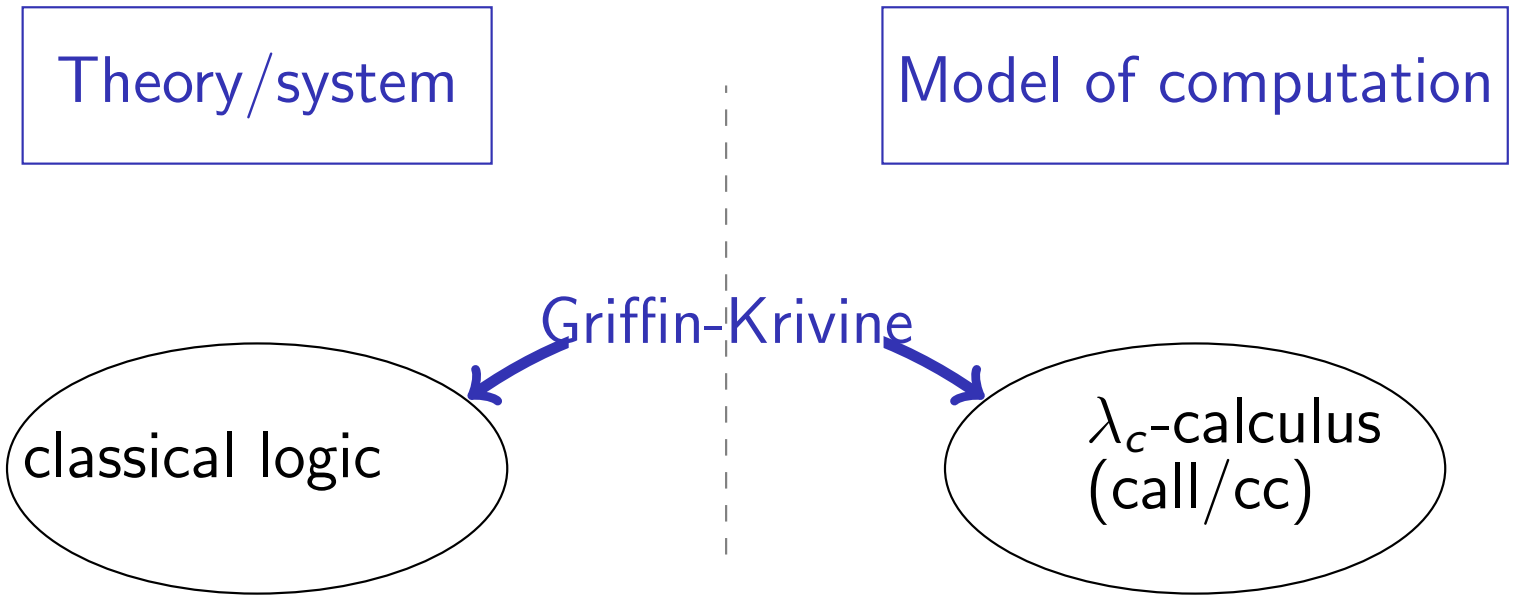

How to realize classical logic?

Classical realizability

We assign to each formula φ

- a **"truth value"** $|\varphi|$ which is a set of programs (λ_c-terms)
- a **"falsity value"** $||\varphi||$ which is a set of stacks

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$$\xi \in |\varphi| \text{ if } \xi \text{ is "incompatible" with every stack in } ||\varphi||$$

We denote by $\perp\!\!\!\perp$ the set of "incompatible processes" $\xi * \pi$.
We write $\xi \Vdash \varphi$ for $\xi \in |\varphi|$.

Classical realizability

We select a set $\mathcal{R}$ of "trustful programs"

$$\text{Theory} := \{\varphi \,:\, \exists \xi \in \mathcal{R}\ \xi \Vdash \varphi\}$$

forms a coherent theory closed by natural deduction

Theorem

$cc \Vdash ((A \Rightarrow B) \Rightarrow A) \Rightarrow A$ (Peirce's Law)

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Any formula in the Theory can be computed in Krivine's abstract machine.

Krivine's machine

$$
\begin{array}{rcll}
tu * \pi & \succ_K & t * u \bullet \pi & (\text{push}) \\
\lambda x.t * u \bullet \pi & \succ_K & t[x := u] * \pi & (\text{grab}) \\
cc * t \bullet \pi & \succ_K & t * k_\pi \bullet \pi & (\text{save}) \\
k_{\pi'} * t \bullet \pi & \succ_K & t * \pi' & (\text{restore})
\end{array}
$$

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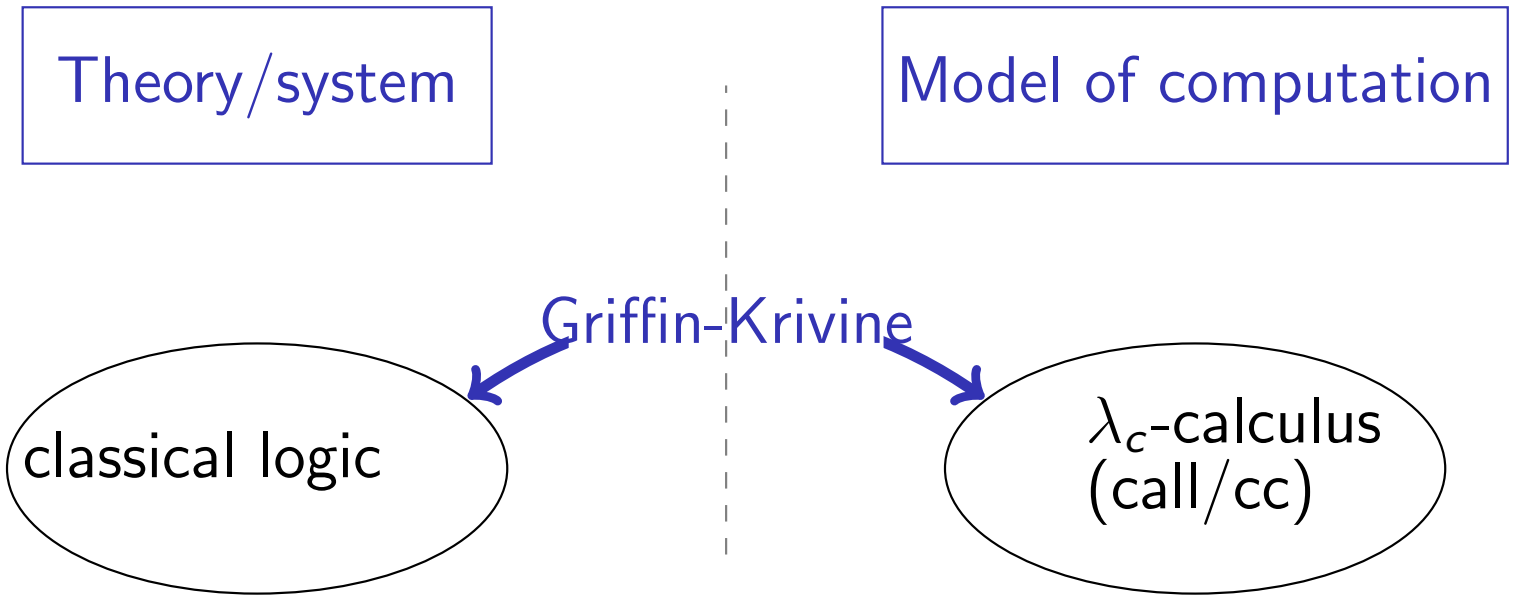

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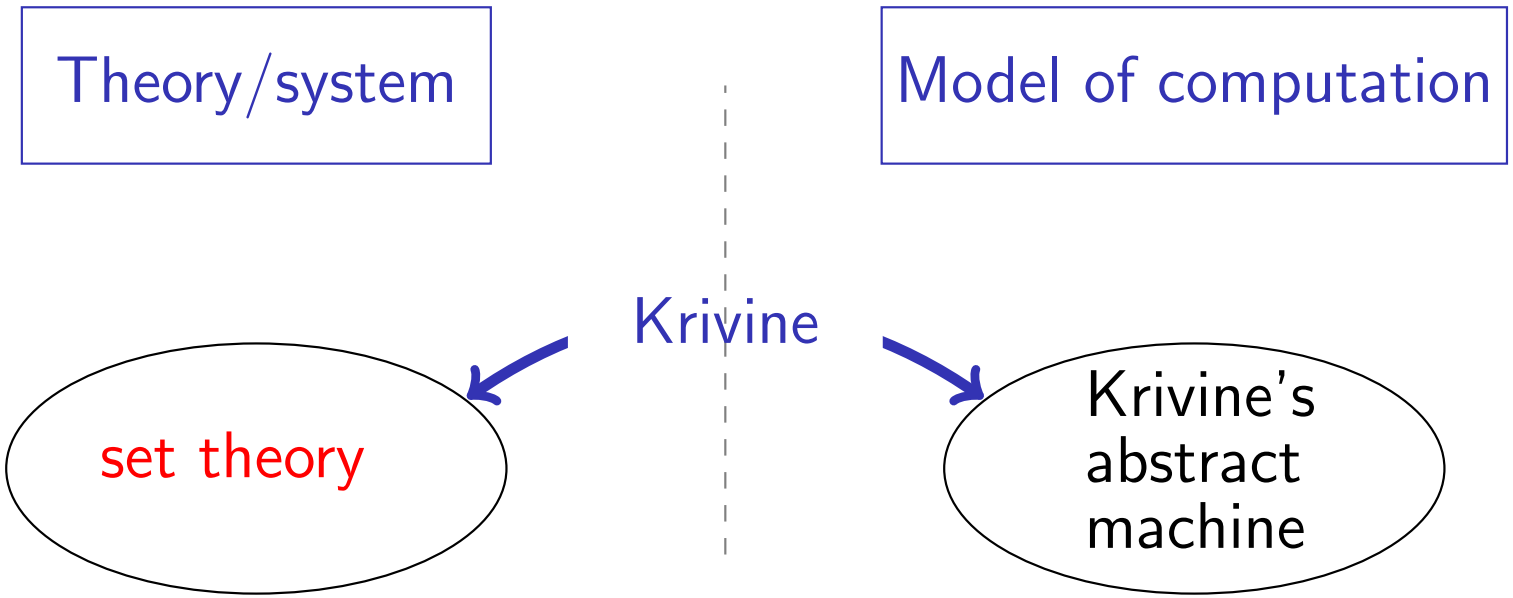

How to realize ZF set theory?

Non extensional set theory

In order to realize the axioms of set theory, we work with a *non-extensional* version of ZF, called $\mathrm{ZF}_{\varepsilon}$.

We consider, two membership relations:

- $\in$ the usual extensional one
- ε a (strict) non-extensional one

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... and two equality relations

- $\simeq$ the usual extensional one $x \simeq y \iff \forall z(z \in x \iff z \in y)$
- $=$ Leibniz identity, i.e. two sets are identical if the satisfy the same formulae

The realizability relation

We define the two truth values $|\varphi|$ and $||\varphi||$.

- no stack is in $||\top||$, every stack is in $||\bot||$
- $||a \notin b|| = \{\pi \text{ stack } ;\ (a, \pi) \in b\}$
- $||a \subseteq b|| = \{\xi \cdot \pi;\ \exists c (c, \pi) \in a \text{ and } \xi \Vdash c \notin b\}$
- $||a \notin b|| = \{\xi \cdot \xi' \cdot \pi;\ \exists c (c, \pi) \in b \text{ and } \xi \Vdash a \subseteq c \text{ and } \xi' \Vdash c \subseteq a\}$
- $||\varphi \Rightarrow \psi|| = \{\xi \cdot \pi;\ \xi \Vdash \varphi \text{ and } \pi \in ||\psi||\}$
- $||\forall x \varphi|| = \bigcup_a ||\varphi[a/x]||$

$\xi \in |\varphi| \iff \forall \pi \in ||\varphi|| (\xi \star \pi \in \perp\!\!\!\perp)$

$\xi \Vdash \varphi$ means $\xi \in |\varphi|$

The realizability model

$$\text{Theory} := \{\varphi \,:\, \exists \xi \in \mathcal{R}\ \xi \Vdash \varphi\}$$

forms a coherent theory closed by natural deduction which contains $\mathrm{ZF}_{\in}$

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Exemple: the identity realizes the axiom of pairing; the axiom of foundation is realized by Turing fixed point combinator; ...

The Axiom of Choice

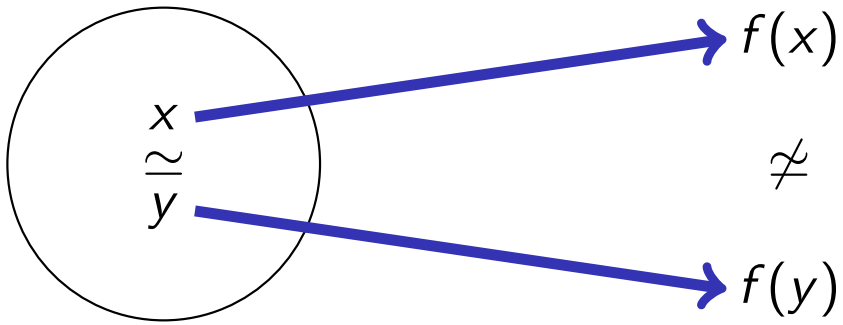
Can we realize the axiom of choice?

Non extensional axiom of choice

We can easily realize a non extensional version of AC, called NEAC
(by many different programs, for instance by the instruction 'quote')

NEAC: existence of a *non-extensional* choice function, i.e.

$$\begin{array}{lcl} x = y & \Rightarrow & f(x) = f(y), \text{ but} \\ x \simeq y & \not\Rightarrow & f(x) \simeq f(y) \end{array}$$



Realizing fragments of AC

Krivine 2003

DC can be realized (using NEAC)

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Fontanella Geoffroy 2020

Uncountable versions of AC can be realized

Realizing the full Axiom of Choice

Theorem (Krivine 2021)

There is a realizability model for the Axiom of Choice (and more)

Realizing the full Axiom of Choice

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but...

... one can show that there is a realizer for AC, but we don't know which one.

Conclusions

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- Realizability is the implementation of this paradigm
- The paradigm and Kleene's realizability were developed as part of a broader research in constructive mathematics
- ...but realizability evolved to include classical logic, set theory and even the axiom of choice

Thank you