Semantic content, the Incompleteness Theorems, and Gödel's reception of Turing's model of computability

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In this talk I will discuss the impact of Turing's formalism independent (Gödel's phrase) conception of computability on Gödel's handling of semantic concepts in the 1930s, through to Gödel's 1946 Princeton Bicentennial Lecture.

Gödel viewed Turing's analysis of computability as paradigmatic, and the impact on his thinking, in 1946 and subsequently, was substantial: mathematically, Gödel's "transfer" of the Turing analysis of computability to the case of provability led to the first formulation of what has come to be known as Gödel's program for large cardinals. In the case of definability this transfer led to the fruitful concept of ordinal and hereditary ordinal definability in set theory.

Philosophically: goal is decidability

Much of Gödel's philosophical work was directed toward the formulation of a view from which the unrestricted application of the Law of Excluded Middle to the entire cumulative hierarchy of sets could be justified.

This means eliminating independence, which always brings undecidability with it.

In the light of the Incompleteness Theorems this means in turn, formalism independence.

(Logical autonomy.)

The claim here is that Gödel's *appropriation* of the Turing analysis lent power and plausibility to his search for a logically autonomous perspective, allowing an overview of logical frameworks, while not being entangled in any particular one of them—*for that is what absolute decidability entails*.

I will claim that the difficulties with establishing—in an absolute sense—the legitimacy of one's computational model, *and the resolution thereof, provided by Turing,* led Gödel to the position: semantic concepts can not be eliminated from our meta-mathematical discourse.

I. The Incompleteness Theorems

Gödel's First version of the First Incompleteness Theorem…

…is heavily semantic: provability is definable in (say) Peano Arithmetic; truth is not definable in Peano Arithmetic, on pain of paradox.

Gödel does not publish this proof, nor does he publish the undefinability of truth theorem.

The 2nd, published version…

…is finitary, or "constructive": the general notion of truth, or the concept of truth in the natural numbers, appears nowhere in the proof.

So Gödel goes to great lengths to eliminate semantic notions from the proof.

However: Gödel must (strongly) represent the primitive recursive functions, i.e. among presenting things like the Fixed Point Theorem and arithmetization, he must prove:

Strong representability of the primitive recursive functions

Theorem 2.1.3 (Theorem V). *Let R be an k-ary primitive recursive relation. Then there is a formula $\phi_R(x_1, \ldots, x_k)$ in the language of S such that:*

(i) $R(n_1, \ldots, n_k)$ *implies* $T \vdash \phi_R(\overline{n_1}, \ldots, \overline{n_k})$

(ii) $\neg R(n_1, \ldots, n_k)$ *implies* $T \vdash \neg\phi_R(\overline{n_1}, \ldots, \overline{n_k})$

Gödel's immediate commentary

Gödel's first comment on his proof is that the theorem is constructive, being that "all existential statements occurring in the proof are based on Theorem V."

Nowhere in the proof of Theorem V is the *full* structure of the natural numbers invoked. Strong representability is proved "piecewise," that is, for a given function defined on a finite segment of the natural numbers. The proof is thus constructively, which at the time was usually employed by Gödel as a synonym for "finitistically," acceptable.

Gödel in later years was interested in defining the limit of finitary reasoning, which for him meant finding a specific proof-theoretic ordinal. A proposal considered in the substantial literature on the limits of finitary reasoning is that *primitive recursive arithmetic* (PRA) should define that limit.

Semantic content?

In the interest of analysing the semantic as well as the finitary content of the First Incompleteness Theorem, consider the proof of Theorem V in the case of, e.g. the exponential function.

I suggest that Gödel is (arguably) appealing to semantic content in the form of the primitive recursive scheme defining the exponentiation function in natural language, for the construction of the formal analogue of the exponentiation function.

Namely…

$m_1^{m_2} = m_3$ holds if and only if there are numbers a_i, $1 \leq i \leq m_2$, such that $a_1 = m_1$, $a_{i+1} = m_1 \cdot a_i$ for $1 \leq i < m_2$, and $a_{m_2} = m_3$. (We suppress the formal symbol for numerals to facilitate ease of reading.) By the above consequence of the Chinese Remainder Theorem, $m_1^{m_2} = m_3$ holds if and only if there are numbers m and n such that $\beta(m, 1, n) = m_1$, $\beta(m, i+1, n) = m_1 \cdot \beta(m, i, n)$ for $1 \leq i < m_2$, and $\beta(m, i+1, n) = m_3$. We may now conclude that $m_1^{m_2} = m_3$ is representable in T. The proof is similar for any primitive recursive function. One simply codes the appropriate sequence by means of the β-function.

Footnote 41: “When this proof is carried out in detail, [the formal object JK], of course, is not defined indirectly with the help of its meaning but in terms of its purely formal structure.”

Is “decidability in a theory” a purely syntactic condition? It is generally thought to be so, on the basis of some mutual understanding of the term “syntactic.” Gödel used the term to mean “devoid of content”:

"The essence of this view is that there is no such thing as a mathematical fact, that the truth of propositions which we believe express mathematical facts only means that (due to the rather complicated rules which define the meaning of propositions, that is, which determine under what circumstances a proposition is true) an idle running of language occurs in these propositions, in that the said rules make them true no matter what the facts are. Such propositions can rightly be called void of content."

(Gibbs lecture in the *Collected Works vol 3*, p. 319.)

Beklemishev observes on Theorem V:

"Gödel's notion of decidability in a theory . . . does not appeal to the 'contentual' meaning (inhaltliche Deutung) of the formulae of the system P. However, one can still see that this notion implicitly appeals to a semantic interpretation of primitive recursive schemes, because the formula φ_R is in fact constructed from a primitive recursive scheme defining R…"

("Gödel's incompleteness theorems and the limits of their applicability. I." Uspekhi Mat. Nauk, 65(5(395)):61–106, 2010.)

In other words, the definition of the formula φ_R, whether in the type-theoretic framework and/or relying on the β-function, draws on the actual primitive recursive definition of the relation R, or more precisely, on its *meaning*.

In that sense this part of the proof may be said to have and/or express an implicit semantic content.

Of course appealing to semantic content in setting up a formal system does not mean that the resulting formal language is to be regarded as contentual.

Build provenance into the system?

The only way this idea can be accommodated in logical practice (to my knowledge) is via the mechanism of an interpretation, that is, the assignment of a (formal) semantics to the formalism in question, and in this way "the formal" is merged with semantic content.

Otherwise there appears to be no notion of syntax on the market, that could accommodate an appeal to the meanings of the pre-theoretic object in question, in the formulation of the syntax—as seems vaguely to happen in strong representability.

The formal analogue of the primitive recursive function, the entire formalism for that matter, is a "genealogical isolate," in the terminology of race theory—stripped of origins, stripped of meaning.

But see…

Erich Grädel: "Semiring Semantics for Logical Statements with Applications to the Strategy Analysis of Games"

Gödel’s later view of semantic concepts

“While a formal system consists only of symbols and mechanical rules relating to them, the meaning which we attach to the symbols is a leading principle in the setting up of the system.”

(1934 Princeton lectures)

Godel to Wang, 1967

"How indeed could one think of expressing metamathematics in the mathematical systems themselves, if the latter are considered to consist of meaningless symbols which acquire some substitute of meaning only through metamathematics?"

In Draft V of Gödel’s paper “Is Mathematics a Syntax of Language” (1951-9) the Beklemishev point is drawn upon again and again: in order to devise the system in question, never mind to ultimately to ascertain its consistency, one needs an available content to begin with, a starting point.

Conventionalism, which Gödel sees as a variant of the syntactic point of view, is here argued against, in particular Gödel argues that conventions regarding symbolic manipulation express or presuppose factual knowledge about symbols, knowledge “which must be known to us already in an empirical attire (i.e. mixed with synthetic facts).” One can adopt the view that conventions are devoid of content in an absolute sense, but:

"If one speaks of conventions and their voidness of content in an absolute sense, this can only mean that they are conventions relative to that body of knowledge which is indispensable for making any linguistic conventions at all."

"These are "unequivocally ascertainable [i.e. true JK] relations between the primitive terms of combinatorics, such as "pair," "equality," "iteration," and they can least of all be eliminated by basing the use of those terms on conventions."

Consistency

However one views the setting up of the formal system, there is, on the opposite side of the spectrum, the issue of the consistency of the entire system, which cannot be derived internally, as a consequence of Gödel's Second Incompleteness Theorem.

"But now it turns out that for proving the consistency of mathematics an intuition of the same power is needed as for deducing the truth of the mathematical axioms, at least in some interpretation. In particular the abstract mathematical concepts, such as "infinite set," "function," etc., cannot be proved consistent without again using abstract concepts, i.e., such as are not merely ascertainable properties or relations of finite combinations of symbols. So, while it was the primary purpose of the syntactical conception to justify the use of these problematic concepts by interpreting them syntactically, it turns out that quite on the contrary, abstract concepts are necessary in order to justify the syntactical rules (as admissible or consistent). . ."

"…the fact is that, in whatever manner syntactical rules are formulated, the power and usefulness of the mathematics resulting is proportional to the power of the mathematical intuition necessary for their proof of admissibility. This phenomenon might be called "the non-eliminability of the content of mathematics by the syntactical interpretation."

(Gibbs lecture, 1953)

II. The *Entscheidungsproblem*

First a little history

Gödel employed the class of *recursive* functions in his landmark 1931 paper.

Recursion was already known: Dedekind (1888, thm 126); Ackermann (1928) separated recursion from primitive recursion, Rosza Péter simplified Ackermann's presentation (1932), coined the term "primitive recursive".

1932: Church introduces the λ-calculus

Church developed the λ-calculus together with Kleene, a type-free and indeed, in Gandy's words, "logic free" model of effective computability, based on the primitives "function" and "iteration."

The phrase "logic free" is applicable only from the point of view of the later 1936 presentation of it, as Church's original presentation of the λ-calculus in his 1932* embeds those primitives in a "deductive formalism," in Hilbert and Bernays's terminology.

Computability in a logic

The attitude in Princeton initially (in the early 1930s) was that computability should be understood in terms of *calculability in a logic*.

(I rely here mainly on the accounts given in Gandy's 1988 "The confluence of ideas in 1936" and Sieg's 1997 "Step by Recursive Step: Church's Analysis of Effective Calculability".)

“Computable in a logic” means…

“And let us call a function F of one positive integer *calculable within the logic* if there exists an expression f in the logic such that $f(\mu) = \nu$ is a theorem when and only when

$F(m) = n$ is true, μ and ν being the expressions which stand for the positive integers m and n.” (Church, 1936)

Circularity?

Such functions F are recursive, according to Church, *if it is assumed that the logic's theorem predicate is recursively enumerable*.

Problem

Church's original presentation of the λ-calculus was found by his students Kleene and Rosser to be inconsistent.

Kleene, Rosser, "The inconsistency of certain formal logics", *Bulletin of the American Mathematical Society*, vol. 41 (1935)

“Not exactly what one dreams of having one’s graduate students accomplish for one.”

---Martin Davis

Kleene: "History of computability in the period 1931-1933"

"When it began to appear that the full system is inconsistent, Church spoke out on the significance of λ-definability, abstracted from any formal system of logic, as a notion of number theory."

Church's Thesis: 1933-4

Church's Thesis begins with Church's verbal suggestions in 1933-4, to identify the λ-definable functions with the *effectively computable* (i.e. human intuitively computable) functions.

Adequacy

By which criteria do we shape, commit ourselves to, or otherwise assess standards of adequacy? This is the problem of *faithfulness*; the problem of what is lost whenever an intuitively given mathematical concept is made exact, or, beyond that, formalized; the problem, in words, of the *adequacy* of our mathematical definitions. It appears to be a philosophical problem rather than a mathematical one, as the idea of "fit" is subject to mathematical proof only very indirectly, if at all.

One coping mechanism: “thesis” language

Church-Turing Thesis: equates the class of intuitively computable number-theoretic functions with the class of functions computable by a Turing Machine. (In its present formulation.)

Weierstrass thesis: the $\varepsilon - \delta$ definition of continuity correctly and uniquely expresses the informal concept.

Dedekind’s thesis: the collection of Dedekind cuts gives the correct definition of the concept “line without gaps”. (Alternatively, that Dedekind gave the correct definition of a natural number.)

Area thesis: the Riemann integral correctly captures the idea of the area bounded by a curve.

Hilbert’s Thesis: “the steps of any mathematical argument can be given in a first order language (with identity).” (Kripke)

“There are theses everywhere.”

Shapiro, “The open texture of computability.” In *Computability—Turing, Gödel, Church, and beyond*.

Evidence for Church's Thesis

Any function that *appeared* to be effectively computable, *was* λ-definable, and conversely.

(Much labor went into showing this!)

Gödel unconvinced…

Gödel: "Thoroughly unsatisfactory."

Church reports the remark in a letter to Kleene dated November 29, 1935. Unsatisfactory is the proposal that the intuitive notion of computability is adequately captured by the computational models discovered to date.

Another line of thought: Herbrand-Gödel Recursion

By 1934, compelled to "make the incompleteness results less dependent on particular formalisms," Gödel introduced in his Princeton lectures of spring 1934, the general recursive or Herbrand-Gödel recursive functions, as they came to be known, defining (along the way) the notion of *"formal system" as consisting of "symbols and mechanical rules relating to them."* Inference and axiomhood were to be witnessed by a finite (primitive recursive) procedure, also the syntax.

Formal system: symbols and mechanical rules

"We require that the rules of inference, and the definitions of meaningful formulas and axioms, be constructive; that is, for each rule of inference there shall be a finite procedure for determining whether a given formula B is an immediate consequence (by that rule) of given formulas A1,...An, and there shall be a finite procedure for determining whether a given formula A is a meaningful formula or an axiom. "

HG calculus continued

The calculus admits forms of recursions that go beyond primitive recursion. Roughly speaking, while primitive recursion is based on the successor function, in the Herbrand-Gödel equational calculus one is allowed to substitute other recursive functions in the equations, as long as this defines a unique function. (For example $f(n) = f(n+1)$ does not define a unique function.)

The (in)adequacy of the HG equational calculus

Gödel, letter to Martin Davis February 15, 1965:

"...I was, at the time of these lectures, not at all convinced that my concept of recursion comprises all possible recursions."

(Indeed, there is little reason to suppose this.)

CT crystallizes in 1935

In Church's lecture on what came to be known as "Church's Thesis" to the American Mathematical Society in 1935, Church uses the Herbrand-Gödel equational calculus as a model of effective computation, i.e. recursiveness in the "new sense," rather than the λ-calculus.

This was preceded by a complex development in which the functions defined in the HG-equational calculus were shown to be the same as the λ-definable functions.

Two approaches in 1936

Church presented two approaches to computability in the AMS lectures and in his subsequent [1936], based on the lectures: Firstly algorithmic (what Gandy had called "logic-free"), based on what is now known as the untyped λ-calculus, i.e. the evaluation of the value f(m) of a function by the step-by-step application of an algorithm—and secondly logical, based on the idea of calculability in a logic.

Hilbert-Bernays' 1939 Grundlagen der Mathematik II

HB present a logical calculus rather than a system of the type of Gödel's [1934]. The essential requirement here is that the proof predicate of the logic is *primitive recursive*.

This effects a precise gain: one reduces effectivity now to *primitive* recursion.

Problem with the logical approach

If effectivity is explained via a logic which is supposed to be given effectively, one must then introduce a new logic, by means of which the effectivity of the initial logic is to be analyzed.

It is natural to assume of the new logic, that it *also* should be given effectively. But then one must introduce a third logic by means of which this effectivity is to be analyzed. And so forth…

Shift in perspective had begun to set in in 1934

Gandy: "… in 1934 the interest of the group shifted from systems of logic to the λ-calculus and certain mild extensions of it…"

Indeed, the Herbrand-equational calculus is not a system of logic per se. Nor is Kleene's 1936 system based on the concept of μ-recursion, a logical calculus; and nor is Post's model of computability presented (also) in 1936 (based on work he had done in the 1920s).

All of these are conceptions of computability given, primarily, mathematically—*but there was no reason whatsoever to believe in their adequacy.*

Confluence not enough

They had *confluence*, i.e. they knew that their various systems were equivalent in the sense of giving the same class of functions. But they lacked a grounding example.

Turing 1936

Rather than calculability in a logic, Turing analyzes effectivity in terms of an informal, fully sharpened, mathematical notion: the concept of a Turing machine.

Turing also solves the *Entscheidungsproblem*!

First proves the unsolvability of the halting problem, which is: given any Turing Machine, can one always decide if it halts or not, on a given input?

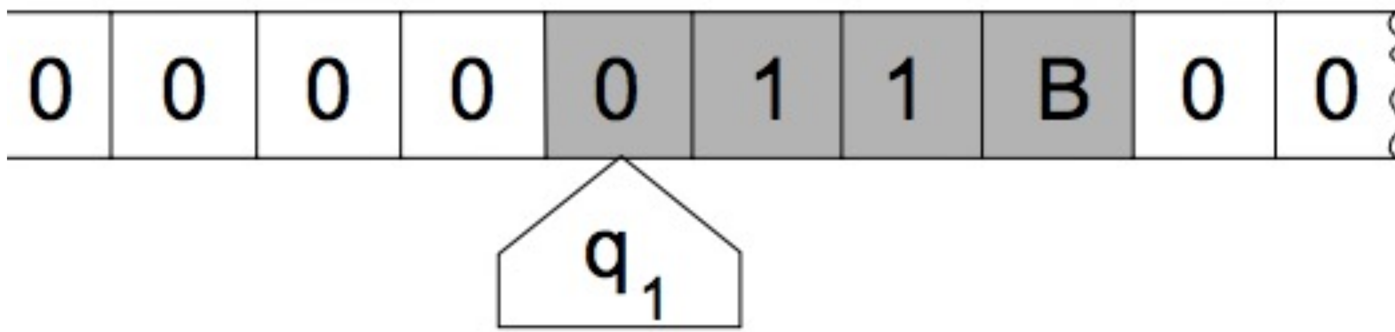
Then shows that provability in FOL is decidable iff the halting problem is solvable.

Church had solved the *Entscheidingsproblem* shortly before…

Turing's Machines

Turing's machine model of computability consists of a tape scanned by a reader, together with a set of simple instructions in the form of quintples.

The analysis consisted of two elements: a conceptual analysis of human effective computation, together with a mathematical precisification of it consisting of rules given by the simple instructions "erase," "print 1," "move left," and "move right."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Reactions

"Turing's computability is intrinsically persuasive but λ-definability is not intrinsically persuasive and general recursiveness scarcely so (its author Gödel being at the time not at all persuaded)."

Kleene, 1981, "The theory of recursive functions, approaching its centennial." *Bull. Amer. Math. Soc. (N.S.), 5(1)*.

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Gödel to Wang

"The resulting definition of the concept of mechanical by the sharp concept of "performable by a Turing machine" is both correct and unique...Moreover it is absolutely impossible that anybody who understands the question and knows Turing's definition should decide for a different concept."

Gödel to Wang continued

"The sharp concept is there all along, only we did not perceive it clearly at first. This is similar to our perception of an animal far away and then nearby. We had not perceived the sharp concept of mechanical procedure sharply before Turing, who brought us to the right perspective."

Grounding

For the logicians of the time, then, the Turing Machine was not just another in the list of acceptable notions of computability—it was the *grounding* of all of them.

Gandy: Turing's isolation from the logical milieu of Princeton was the key

"It is almost true to say that Turing succeeded in his analysis because he was not familiar with the work of others. . . The bare hands, do-it-yourself approach does lead to clumsiness and error. But the way in which he uses concrete objects such as exercise book and printer's ink to illustrate and control the argument is typical of his insight and originality. Let us praise the uncluttered mind. "

Generality of the Incompleteness Theorems

A precise notion of formal system was needed for settling the question, taken up by Gödel himself in his 1931 paper on Incompleteness, whether those theorems are completely general, that is, whether they apply to any formal system containing arithmetic, and not just Principia and systems related to it. Gödel was careful to say at the end of his 1931 paper that this had not been shown. The issue lingered for some time after the Incompleteness Theorems had been published.

Turing analysis resolves the issue in Gödel's view

"In consequence of later advances, in particular of the fact that, due to A. M. Turing's work, a precise and unquestionably adequate definition of the general concept of formal system can now be given, the existence of undecidable arithmetical propositions and the non-demonstrability of the consistency of a system in the same system can now be proved rigorously for *every* consistent formal system containing a certain amount of finitary number theory." Gödel, 1965, Postscriptum to his 1934 lectures.

. . . Turing's work gives an analysis of the concept of "mechanical procedure" (alias algorithm or computation procedure or "finite combinatorial procedure"). This concept is shown to be equivalent with that of a "Turing machine." A formal system can simply be defined to be any mechanical procedure for producing formulas, called provable formulas. For any formal system in this sense there exists one in the [usual] sense that has the same provable formulas (and likewise vice versa). . .

"That my [incompleteness] results were valid for all possible formal systems began to be plausible for me (that is since 1935) only because of the Remark printed on p. 83 of 'The Undecidable' . . . But I was completely convinced only by Turing's paper. "

Gödel, letter to Kreisel (printed in Odifreddi.)

Footnote to 1934 lectures

"In my opinion the term "formal system" or "formalism" should never be used for anything but this notion [i.e. a mechanical procedure in the sense of the Turing Machine JK].", *Collected Works. I: Publications 1929–1936*, p. 195

“All the work described in Sections 14.3-14.977 [work of Church etc JK] was based on the mathematical and logical (and not on the computational) experience of the time. What Turing did, by his analysis of the processes and limitations of calculations of human beings, was to clear away, with a single stroke of his broom, this dependence on contemporary experience, and produce a characterization which—within clearly perceived limits—will stand for all time. ”

Now to the *Entscheidingsproblem,* and why Gödel doesn't offer a solution to it

Formulated in its standard form in Hilbert and Ackermann's 1928, it asks whether there is an algorithm for deciding validity for first order logic, i.e. if there is an algorithm which decides in a yes or no manner for any first order statement P, whether it is valid or not.

Gödel's Completeness Theorem equates first order validity with the existence of a finite proof, so the *Entscheidungsproblem* is equivalent to the question whether, for any recursively axiomatized first order theory, there is an algorithm for deciding whether a first order statement in the language of the theory follows from the axioms.

Put another way…

Given that the proof predicate for e.g. first order Peano Arithmetic is Σ_1 (or recursively enumerable, i.e. r.e.), the *Entscheidungsproblem* asks whether the provability predicate for e.g. first order Peano is not only r.e. but recursive.

Herbrand, 1929

The decision problem is "…the most important of those, which exist at present in mathematics."

The unsolvability of the Decision Problem (*Entscheidungsproblem*) was established independently by Church and Turing in 1936, using conceptually distinct methods.

Church's Thesis is used here: the notion of "algorithm" at issue in the *Entscheidungsproblem* is adequately represented by the mathematical notion of "recursive."

Many logicians have remarked on the close relationship between the *Entscheidungsproblem* and incompleteness, e.g. R. Gandy. “Thus Gödel’s result,” as Gandy would write in his brilliant survey paper “The confluence of ideas in 1936” , “meant that it was almost inconceivable that the *Entscheidungsproblem* should be decidable: a solution could, so to speak, only work by magic.”

On magic

If the *Entscheindingsproblem* were **decidable**, then we would know that the algorithm always gives an answer, however we could not **prove** that it always gives an answer.

This seems odd.

In Kripke's view Gödel's 1931 actually solves the *Entscheidungsproblem*!

In his view, Gandy's remark (on "magic") is "much too weak," as the unsolvability of the *Entscheidungsproblem* is a corollary of Gödel's 1931 paper, in particular of Theorem IX:

"Gödel's Theorem IX clearly directly implies Turing's result that the *Entscheidungsproblem* is not decidable on one of his machines, since we can simply add an axiomatization of the operation of the machine to his basic system."*

* "The Church-Turing "Thesis" as a special corollary of Gödel's completeness theorem." In *Computability: Gödel, Church, and Beyond.*

The argument, roughly, is as follows.

First adopt Kripke’s (in his terminology) a “logical” view of computation, namely that computation should be regarded as a special form of mathematical argument:

"My main point is this: computation is a special form of mathematical argument. One is given a set of instructions, and the steps in the computation are supposed to follow—follow deductively— from the instructions as given.

It is in this sense…that I am regarding computation as a special form of deduction, that I am saying I am advocating a logical orientation to the problem."

Hilbert's Thesis

The second ingredient Kripke relies on for his claim that the negative solution of the Entscheidungsproblem is a corollary of Gödel's Theorem IX, is what he calls "Hilbert's Thesis," namely the idea that "the steps of any mathematical argument can be given in a language based on first-order logic (with identity)."

Kripke will use Hilbert's Thesis together with Gödel's Completeness Theorem to infer that any valid computation, if viewed as a valid deduction, is provable in any of the standard first-order formal systems.

Gödel’s Theorem IX of *1931*

“For any of the formal systems mentioned in theorem VI,” there are undecidable problems of a first order form, i.e. **first order** formulas for which neither validity not the existence of a counterexample is provable.

The proof of the undecidability of the *Entscheidingsproblem* from theorem IX:

1. Suppose the algorithm α solves the Entscheidungsproblem.
2. Let Σ be strong enough to prove Gödel's Theorem X and other necessary things. (E.g. Σ can be ZF^-: enough describe the semantics.)
3. Let $A(x)$ say that the algorithm α halts and says that the algorithm says that x is valid. (Computability as a form of deduction + Hilbert's thesis.)
4. For all ψ, $\Sigma \vdash A(\psi)$ or $\Sigma \vdash \neg A(\psi)$ because α always gives an asnwer.
5. We may assume Σ includes $\forall x(A(x) \equiv Val(x))$.
6. By Gödel's Theorem IX, together with the First Incompleteness Theorem, there is ϕ_Σ such that ϕ_Σ is satisfiable (has a model) but $\Sigma \not\vdash$ "ϕ_Σ is satisfiable" (Gandy points this out too.)
7. $\Sigma \not\vdash Sat(\phi_\Sigma)$, by the above. ($\Sigma$ can express Sat, i.e. that the negation of ϕ is not satisfiable.)
8. $\Sigma \not\vdash \neg Val(\neg\phi_\Sigma)$, by logic.
9. $\Sigma \not\vdash \neg A(\neg\phi_\Sigma)$, by logic.
10. $\Sigma \vdash A(\neg\phi_\Sigma)$, because α gives always an answer.
11. $\Sigma \vdash Val(\neg\phi_\Sigma)$, a contradiction with the assumption that ϕ_Σ is satisfiable.

A repeat of the hesitation to publish the semantic version of the f.i.t. in 1930?

The above argument requires acceptance of Church's Thesis.

It also requires the semantic notion of truth in the standard model. (In order to prove Theorem X, from which theorem IX follows.)

Reasons…

Does this speak to Gödel's deference to the anti-metaphysical atmosphere of the times? Or does it speak to a deeper anti-truth stance, however temporary?

Assumption: Gödel knew this argument!

A disagreement with Kripke

"...why didn't Gödel ...regard Theorem IX as such a proof? One problem in the argument I have given that Theorem IX is such a proof is its free use of the notion of truth... **However, it seems very unlikely that Gödel, at least, would have regarded that as a questionable part of the argument.** What seems most likely lacking is an appropriate analog of Church's thesis."

III. Gödel’s 1946 Lecture, and the appearance of ***truth as a primitive notion***

“Tarski has sketched in his lecture the great importance (and I think justly) of the concept of general recursiveness (or Turing computability). It seems to me that this importance is largely due to the fact that with this concept one has succeeded in giving an absolute definition of an interesting epistemological notion, i.e. one not depending on the formalism chosen.”

"In all other cases treated previously, such as demonstrability or definability, one has been able to define them only relative to a given language, and for each individual language it is clear that the one thus obtained is not the one looked for."

E.g. being definable in set theory is not definable. "Take the least undefinable ordinal…"

"This, I think, should encourage one to expect the same thing to be possible also in other cases (such as demonstrability or definability). It is true that for these other cases there exist certain negative results, such as the incompleteness of every formalism . . . But close examination shows that these results do not make a definition of the absolute notions concerned impossible under all circumstances, but only exclude certain ways of defining them, or at least, that certain very closely related concepts may be definable in an absolute sense."

What are we aiming at?

Gödel: Intuitive concept of definability to be made precise: “Comprehensibility by our mind.”

Provability

"Let us consider, e.g., the concept of demonstrability. It is well known that, in whichever way you make it precise by means of a formalism, the contemplation of this very formalism gives rise to new axioms which are exactly as evident and justified as those with which you started, and that this process of extension can be extended into the transfinite. So there cannot exist any formalism which would embrace all these steps; but this does not exclude that all these steps . . . could be described and collected in some non-constructive way. In set theory, e.g., the successive extensions can be most conveniently represented by stronger and stronger axioms of infinity."

Truth as a primitive notion

"…It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from the present [ZFC] axioms plus some **true** assertion about the largeness of the universe of all sets."

In brief…

Some suitable hierarchy of large cardinal assumptions should replace the hierarchy of formal systems generated by, e.g., the addition of consistency statements to set theory, i.e., passing from ZFC to ZFC+Con(ZFC) and then iterating this; or the addition of a satisfaction predicate for the language of set theory, then considering set theory in the extended language, and iterating this.

First proposal for definability: Constructibility

- Constructible sets (L):

$$
\begin{aligned}
L_0 &= \emptyset \\
L_{\alpha+1} &= \mathrm{Def}(L_\alpha) \\
L_\nu &= \bigcup_{\alpha<\nu} L_\alpha \text{ for limit } \nu
\end{aligned}
$$

Gödel's two notions of definability

- Two canonical inner models:
  - Constructible sets (L)
    - Model of ZFC
    - Model of GCH
    - *Definable, but in analogy with the Kleene T-predicate this does not provide a mechanism to transcend L*
  - Hereditarily ordinal definable sets (HOD)
    - Model of ZFC
    - CH? – independent
    - *Definable, but in analogy with the Kleene T-predicate this does not provide a mechanism to transcend HOD*

Summary

3 “epistemological” notions: computability, provability, definability.

Each come with their own paradoxes.

Gödel wants to adapt the Turing analysis* of computability to the cases of provability and definability.

*find a formalism independent characterisation of the concept

Back to the future

Gödel's question about generalizing the Turing analysis to the cases of provability and definability in his 1946 Lecture plunges us into a set of issues that are very similar to those faced by the logicians of the 1930s, prior to Turing.

Truth

- Is generally taken as a primitive notion in Gödel’s philosophical writings from 1936 (i.e. post-Turing) onwards.
- Axioms force themselves on us as being true etc…

IV. The Machine metaphor

1934 lecture

"When I first published my paper about undecidable propositions the result could not be pronounced in this generality, because for the notions of **mechanical procedure** and of formal system no mathematically satisfactory definition had been given at the time. This gap has since been filled by Herbrand, Church and Turing. The essential point is to define what a procedure is. Then the notion of formal system follows easily…"

What was so compelling about the machine metaphor?

The "automated condition" is often presented today in a negative light, as a condition of fatal passivity.

Hannah Arendt warns in *The Human Condition* that advancements in automation enabled by the industrial revolution and invention the steam engine, could result "in the deadliest, most sterile passivity history has ever known."

But this is new:

Descartes Animals as machines

L'homme machine, Julien Offray de la Mettrie, 1748: the soul is to be "clearly an enlightened machine."

The Human Motor, Anson Rabinbach, 1900: the "human motor" as a key metaphor of the industrial era

Manifesto del Futurismo, 1909, Marinetti: glorification of the machine. "A roaring motor car which seems to run on machine-gun fire, is more beautiful than the Victory of Samothrace."

Marinetti to Severini: "try to live the war [WWI JK] pictorially, studying it in all its marvelous mechanical forms."

Letter, Raymond Duchamp-Villon, 1913: "The power of the machine asserts itself and we can scarcely conceive living beings anymore without it."

Reciprocity between metaphors: body-as-machine, machine-as-body.

Unique Forms of Continuity in Space
Boccioni, 1913

Question

Did the idealization of the machine contribute to the acceptance of the Turing Machine as adequate for human effective computability?


ROGER DUBUIS

Thank you!

IV. Implementation

Study the degree to which canonical mathematical structures are **entangled** with (sensitive to) their underlying logic (or formalism), or alternatively persistent under permutation of these, i.e. **formalism free**.

Recall…

By a formalism, or a *logic,* we mean a combination of a list of symbols, commonly called a signature, or vocabulary; rules for building terms and formulas, a list of axioms, rules of proof, and finally a definition of the associated semantics.

Definition

- With this concept of formalism we associate *formalism freeness* with the suppression of any or all of the above aspects of a logic, except semantics.
- Of course vocabulary in the *informal, natural language sense,* detached from any formalism, is always a feature of the practice and in that sense is not suppressed. The difference between the use of names in the mathematician's natural language and the vocabulary of a formal language.

Tarski

A class of structures in a finite relational language is universally (Pi_1) axiomatizable if and only if it is closed under isomorphism, substructure and if for every finite substructure B of a structure A, $B \in K$ then $A \in K$.

Choice of primitives

"The mutual interpretability between classical geometries and fields theories can only be treated as a functor preserving model completeness by a very careful choice of the primitives for the geometry (particularly) for the order case."

Observation in Manders 1984, "Interpretations and the model theory of the classical geometries."

Joint work with M. Magidor and J. Väänänen

Recall Constructibility

- Constructible sets (L):

$$
\begin{aligned}
L_0 &= \emptyset \\
L_{\alpha+1} &= \mathrm{Def}(L_\alpha) \\
L_\nu &= \bigcup_{\alpha<\nu} L_\alpha \text{ for limit } \nu
\end{aligned}
$$

Gödel's alternate definition of L making no reference to definability

1. $\mathfrak{F}_1(X,Y) = \{X,Y\}$
2. $\mathfrak{F}_2(X,Y) = E \cdot X = \{(a,b) \in X \mid a \in b\}$
3. $\mathfrak{F}_3(X,Y) = X - Y$
4. $\mathfrak{F}_4(X,Y) = X \restriction Y = X \cdot (V \times Y) = \{(a,b) \in X \mid b \in Y\}$
5. $\mathfrak{F}_5(X,Y) = X \cdot \mathfrak{D}(Y) = \{b \in X \mid \exists a (a,b) \in Y\}$
6. $\mathfrak{F}_6(X,Y) = X \cdot Y^{-1} = \{(a,b) \in X \mid (b,a) \in Y\}$
7. $\mathfrak{F}_7(X,Y) = X \cdot \mathfrak{Cnv}_2(Y) = \{(a,b,c) \in X \mid (a,c,b) \in Y\}$
8. $\mathfrak{F}_8(X,Y) = X \cdot \mathfrak{Cnv}_3(Y) = \{(a,b,c) \in X \mid (c,a,b) \in Y\}$

C($\mathcal{L}$*)

- $\mathcal{L}$* any logic. We define C($\mathcal{L}$*):

$$
\begin{aligned}
L'_0 &= \emptyset \\
L'_{\alpha+1} &= \mathrm{Def}_{\mathcal{L}^*}(L'_\alpha) \\
L'_\nu &= \bigcup_{\alpha<\nu} L'_\alpha \text{ for limit } \nu
\end{aligned}
$$

- C($\mathcal{L}$*) = the union of all L'_α

- Myhill-Scott result: Hereditarily ordinal definable sets (HOD) can be seen as the constructible hierarchy based on second order logic (in place of first order logic):

$$
\begin{array}{lcl}
L'_0 & = & \emptyset \\
L'_{\alpha+1} & = & \mathrm{Def}_{SOL}(L'_\alpha) \\
L'_\nu & = & \bigcup_{\alpha<\nu} L'_\alpha \text{ for limit } \nu
\end{array}
$$

Two extremes

If $\mathcal{L}^*$ is first order logic, we get L.

If $\mathcal{L}^*$ is second order logic, we get HOD.

If V=L

- If V=L, then V=HOD=Chang's model=L.
- If there are uncountably many measurable cardinals then AC fails in the Chang model. (Kunen.)

Outcomes:

- For a variety of logics $C(\mathcal{L}^*)=L$
  - Gödel's L is very robust, not limited to first order logic
- For a variety of logics $C(\mathcal{L}^*)=HOD$
  - Gödel's HOD is robust, not limited to second order logic
- For some logics $C(\mathcal{L}^*)$ is a potentially interesting new inner model.

Robustness of L

- $Q_1 x\varphi(x) \Leftrightarrow \{a : \varphi(a)\}$ is uncountable
- $C(\mathcal{L}(Q_1)) = L$.
- In fact: $C(\mathcal{L}(Q_\alpha)) = L$, where
  - $Q_\alpha x\varphi(x) \Leftrightarrow |\{a : \varphi(a)\}| \geq \aleph_\alpha$
- Other logics, e.g.

  weak second order logic, ``absolute” logics, etc.

Robustness of L (contd.)

- A logic $\mathcal{L}^*$ is absolute if ``$\varphi \in \mathcal{L}^*$'' is $\Sigma_1$ in $\varphi$ and ``$M \models \varphi$'' is Δ_1 in M and φ in ZFC.
  - First order logic
  - Weak second order logic
  - $\mathcal{L}(Q_0)$: ``there exists infinitely many
  - Finitary fragment of $\mathcal{L}_{\omega_1\omega}$, $\mathcal{L}_{\infty\omega}$: infinitary logic
  - Finitary fragment of $\mathcal{L}_{\omega_1 G}$, $\mathcal{L}_{\infty G}$: game quantifier logic

Theorem

- For all F: $C(\mathcal{L}^{2,F})$=HOD
- Third order, fourth order, etc gives HOD.

Observations: avoiding L

- $C(\mathcal{L}_{\omega_1\omega}) = L(\mathsf{R})$

(every formula in lhs can be coded by a real; every real can be coded by a formula of lhs.)

- $C(\mathcal{L}_{\infty\omega}) = V$ (same as above, but for sets)

Other Generalized Quantifiers

$Q_1^{MM}xy\varphi(x,y) \Leftrightarrow$ there is an uncountable X such that $\varphi(a,b)$ for all a,b in X

- Can express Suslinity of a tree. (No uncountable branches, no uncountable antichains)
- Is countably compact (i.e. w.r.t. countable theories) if V=L. L-Skolem down to $\aleph_1$. Can be badly incompact.

- $Q_0^{cf}xy\varphi(x,y) \Leftrightarrow \{(a,b) : \varphi(a,b)\}$ is a linear order of cofinality ω
  - Fully compact extension of first order logic. (Whatever the size of the vocabulary, if a theory of this logic is finitely consistent, then it is consistent.) L-Skolem down to $\aleph_1$.

C*

- $Q_0^{cf}xy\varphi(x,y)$ we denote "C*"

- C* knows which ordinals have cofinality ω

  in V

Hitting L

$C(\mathcal{L}(Q_1^{MM})) = L$, in the presence of large cardinals. (L thinks that $\mathcal{L}(Q_1^{MM})$ is FO, in spite of it's being very far from FO in the sense of being badly incompact.)

Proof: if there is an uncountable homogeneous set in V then there is one in L. Uses an argument based on the indiscernibles given by large cardinals.

Avoiding L

- $C^* \neq L$, in the presence of large cardinals. (Surprise is that cofinality is very close to FO logic in exhibiting a high degree of compactness.)
- Proof: if α is regular in L and cofinality of α is $>\omega$ in V, we can express this in C^*. But then α belongs to the set of canonical indiscernibles, i.e we can define $0^\#$ in C^*.

More theorems

- If V=C* then continuum is at most $\omega_{2,}$ and there are no measurable cardinals.
- A real is always constructed on levels of rank less than ω_2 in V.
- Second part is like the Scott proof.

Proof of V=C* implies continuum is at most ω_2.

- Condensation argument.
- If *r* is a real, then *r* is in some X $\prec$ C*. By starting with countable and building a chain of length ω_1 we can assume wlog that X (has cardinality ω_1 and) "knows" about cofinality ω. Need witnesses both for cofinality ω and for cofinality greater than ω. In latter case we change the higher cofinalities to cofinality ω_1 by the chain argument. (Problem was that Mostowski collapse doesn't necessarily preserve cofinalities **in V**.)
- Then X $\cong$ L'_α for some α, $\alpha<\omega_2$
- Thus there are at most ω_2 reals.
- Consistently (Namba forcing): exactly ω_2 reals.

$V=C^*$ implies that there are no measurable cardinals.

- Suppose $i:V \to M$, κ first ordinal moved, M closed under κ-sequences.
- $(C^*)^M=C^*$, since M and V have the same ω-cofinal ordinals (since they have the same ω-sequences).
- So M=V.
- $i:V \to V$, κ first ordinal moved
- Contradiction! (By Kunen.)
- Smaller large cardinals are consistent with V=L, hence with $V=C^*$.

More theorems

If there is a Woodin cardinal, then ω_1 is inaccessible in C*. (Stationary tower forcing. Gives an embedding into a model which is closed under ω sequences, which moves ω_1 to the Woodin cardinal. Then $(C^*)^M = (C^*_{<\lambda})^V$.

Actually Mahlo. And ω_2 of V is WC in C*.

Generic absoluteness

- Suppose there is a proper class of Woodin cardinals. Then:
- Truth in C* is forcing absolute and independent of α. (Stationary tower forcing.)
- Cardinals $>\omega_1$ are all indiscernible for C*. Another STF.
- Is CH true in C*? *This is forcing absolute.*

Large cardinals in these inner models

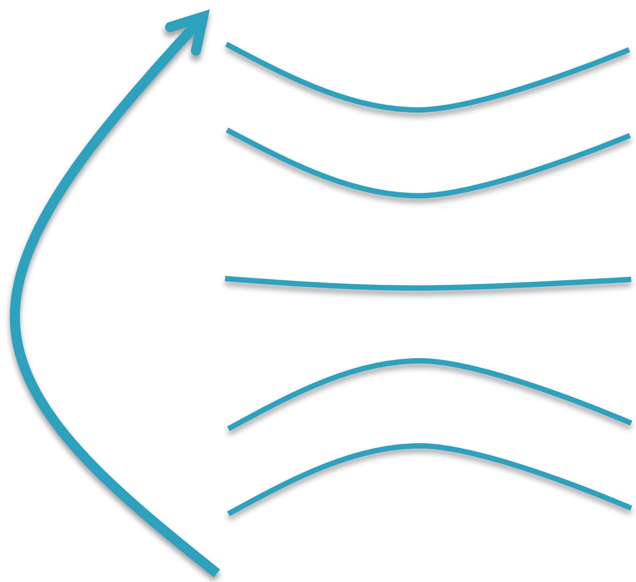
- Let C(aa) be like C* but use the *stationary logic* instead of cofinality quantifier.
- Stationary logic has a quantifier ``for a club of countable subsets X'' $\varphi(X)$.
- MM^{++} implies uncountable cardinals are measurable in C(aa). (Magidor)

In fact, this is a general schema…

R(L, O)

$HOD = C(SOL)$

One application of power-set



Hierarchy of generalized quantifiers.

$L = C(FOL)$

Bagaria, J., Väänänen, J**. "On the Symbiosis between Model-theoretic and Set-theoretic Properties of Large Cardinals,"** in preparation.

Kennedy, J**. "On Formalism Freeness: Implementing Gödel's 1946 Princeton Bicentennial Lecture,"** *Bulletin of Symbolic Logic, volume 19, issue 3, September 2013, 351-393*.

Kennedy, J**. "On the Logic without Borders Point of View,"** *Logic without Borders,* de Gruyter, 2014**.**

Kennedy, J. **Turing, Gödel and the "Bright Abyss,"** Proceedings of the AT100 Symposium at BU, to appear.

Kennedy, J., Magidor, M., Väänänen, J. **"Inner Models from Extended Logics,"** preprint

Väänänen, J**. "Abstract logic and set theory, I: Definability,"** Logic Colloquium '78, pages 391-421, North-Holland, 1979, M.Boffa, D.van Dalen and K.McAloon, editors.

Another data point: “Unreasonable” effectiveness of semantic methods:

- Theorem (Väänänen-Vardi, Gödel, Parikh): Given a concept of provability in predicate logic, there is no recursive function f such that for all φ that are valid, if the length of the (semantic) proof (in set theory) of validity of φ is n, then the length of the predicate logic proof of φ is at most f(n).

- P. 121
- But if I am right, the situation surveyed in the previous section is irremediable—there is no viable way for the classical picture to assign stable contents to a range of familiar predicates (it is thus doomed, on my view, to remain merely a picture forever).
- 126 It so happens that, if the inferential structures of a domain can be organized in axiomatic fashion, then logical connections such as modus ponens and universal instantiation can seem as if they rep- resent the central inferential relationships within the subject (I regard this point of view as erroneous: even in an axiomatic system, the dominant inferential structures of classical mechanics are closely tied to more specialized forms of reasoning and the particular features of differential equations). This logic-centered focus has occasioned a rather odd historical development. Many philosophers and logicians in the 1920s became con- vinced that quite general problems in philosophy could be profitably addressed by considering the behaviors of schematic or toy axiomatic systems (which were invariably dubbed T and T0, hence my syndrome's label).
- PAGE 127!!!!!

Pierce's paradox

- Overlapping languages
- Mark Wilson's example in chapter 3, p. 116

Fortunately, we do not need to contend with these ramifications now, but only bear them gently in mind as we forge ahead. However, it helps to be prepared for the following eventuality: a particular predicate "P" has adequately established its practical credentials, but our present conception of its directive core has become shaken. Somehow we must find a replacement rationale for threading its satellite standards of correctness together, a process I shall later call "putting a new picture to it." We'll find that such occasions arise fairly frequently in the career of many descriptive predicates.

Precedent

"Although Carol Karp appreciated recursive function theory, she disliked proofs which involved codings and systems of notations. In her work on infinitary set theory she noticed that infinitely long formulae sometimes allowed her to circumvent notations…She discovered that by varying the logic in the system one could get a host of results about recursion theory and its extensions; furthermore it could be done without any ad hoc notations. Unfortunately, she only had time to work out some of the details for fragments of infinitary languages of the form Law (i.e., finite-quantifier infinitary languages)."

It was her research on the infinitely long formal proofs that led Karp to the concept of L-R.E. on A. However, it is clear that the actual structure of the proofs is irrelevant, for all that is ever used is the consequence relation. Thus, for the purpose of discussing extensions of recursion theory, it does not make much sense to dwell too much upon the axioms and rules of inference. Consistency properties are a natural way of getting all the benefits of completeness while, at the same time, avoiding formal proofs. (Lopez-Escobar, *Introduction, Infinitary logic: in memoriam Carol Karp*)

"…It is certainly impossible to give a…decidable characterization of what an axiom of infinity is; but there might exist, e.g., a characterization of the following sort: An axiom of infinity is a proposition which has a certain (decidable) formal structure and which in addition is true. Such a concept of demonstrability might have the required closure property, i.e., the following could be true: Any proof for a set-theoretic axiom in the next higher system above set theory (i.e., any proof involving the concept of truth which I just used) is replaceable by a proof from such an axiom of infinity."

“…while formalization is the key tool for the general foundational analysis and has had significant impact as a mathematical tool, there are specific problems in mathematical logic and philosophy where ‘formalism-free’ methods are essential.”

---J. Baldwin, *op cit*

Philosophical commitments

- The actual content of mathematics goes beyond any formalization. C. McClarty refers to this as "expansive intuitionism," his term for Poincaré's reaction (or counterreaction) to formalism:

"…for Poincaré formal proof never itself yields knowledge—neither for the inventor nor for the student. Mathematical knowledge comes from mathematical reasoning, which is contentual and not formal." Mclarty, *Poincaré: Mathematics & logic & intuition*